OPTICS IN HYPERBOLIC SPACE*

BY

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1. Introduction

With Riemann we define the metric of H-space† by

$$ds^2 = \frac{dx_1^2 + dx_2^2 + dx_3^2}{\left[1 - \frac{x_1^2 + x_2^2 + x_3^2}{4R^2}\right]^2} = 16R^2 \frac{d\sigma^2}{\lambda^2}, \tag{1}$$

where

$$r^2 = x_1^2 + x_2^2 + x_3^2, \quad d\sigma^2 = dx_1^2 + dx_2^2 + dx_3^2, \quad \lambda = 4R^2 - r^2. \tag{2}$$

H-straights are defined by

$$\delta \int ds = 0. \tag{3}$$

In order to have a model of this space in which we can see the figures employed we may regard the x_1, x_2, x_3 as rectangular cartesian coördinates. Then the images of points of H_3 are points within the e-sphere $\lambda = 0$ which we call the λ-sphere. In this model H-straights are e-circles cutting $\lambda = 0$ orthogonally.

It is convenient to introduce new variables

$$z_i = \frac{4R^2}{\lambda} x_i, \quad i = 1,2,3; \quad z_4 = R(\mu/\lambda), \tag{4}$$

where

$$\mu = x_1^2 + x_2^2 + x_3^2 + 4R^2.$$

H-planes are defined by a linear relation

$$a_1 z_1 + a_2 z_2 + a_3 z_3 + a_4 z_4 = 0.$$

In the model they are e-spheres cutting $\lambda = 0$ orthogonally. The intersection of two H-planes are H-straights. H-planes through the origin O are also e-planes and the same is true of straights. The H-angle between two curves or surfaces or a curve and a surface is the same as the corresponding angle

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† For H- read hyperbolic; for e- read euclidean.

in the model. Figures may be moved about freely in H-space as in e-space. The coördinates z satisfy

$$\{z^2\} = z_1^2 + z_2^2 + z_3^2 - z_4^2 = -R^2. \tag{5}$$

Besides the points so far considered we have certain ideal points, viz., those lying on the λ-sphere; they are at an infinite distance away from any ordinary point. Their z coördinates satisfy the relation

$$\{z^2\} = 0. \tag{6}$$

The four planes $z_1=0, z_2=0, z_3=0, z_4=0$ form a tetrahedron, the plane $z_4=0$ being imaginary. It may be represented diagramatically by Fig. 1. The vertex A_k is opposite $z_k=0$. All straights perpendicular to $z_k=0$ meet in the vertex A_k. The displacement

$$z_1' = z_1, \quad z_2' = z_2 \cosh + z_3 \sinh \theta,$$
$$z_3' = z_2 \sinh \theta + z_3 \cosh \theta, \quad z_4' = z_4$$

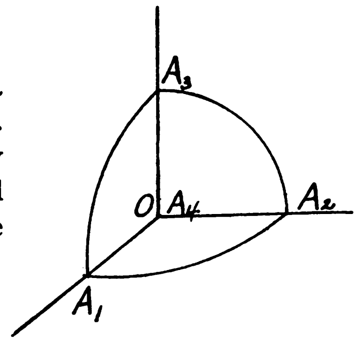

FIG. 1

defines a rotation θ about A_1. It leaves the plane $z_1=0$ unaltered, a figure in this plane being merely moved into a congruent figure.

2. Reflection and refraction on a plane surface

We shall suppose that the path of a ray of light in a heterogeneous medium satisfies

$$\delta \int n ds = 0, \tag{1}$$

where n, the index, is a function of the coördinates x. When $n=$constant this becomes

$$\delta \int ds = 0, \tag{2}$$

i.e., the path of a ray of light is a straight.

Consider two media of indices n, n' separated by an H-plane. A ray issuing from A arrives at A'. It is easy to see that the path lies in a plane normal to the boundary. We suppose it lies in the x, y plane. We must choose B so that $nAB+n'BA'$ or $s=np+n'p'$ is a minimum. Then

$$\frac{ds}{dx} = n\frac{dp}{dx} + n'\frac{dp'}{dx} = 0. \tag{3}$$

Let $CB=x$, $BC'=x'$, $x+x'=c$, a constant, and therefore $dx+dx'=0$. Then

$$\cosh(p/R) = \cosh(a/R)\cosh(x/R), \quad \cosh(p'/R) = \cosh(a'/R)\cosh(x'/R)\,;$$

therefore

$$\frac{dp}{dx} = \cosh(a/R)\frac{\sinh(x/R)}{\sinh(p/R)}, \qquad \frac{dp}{dx} = -\cosh(a'/R)\frac{\sinh(x'/R)}{\sinh(p'/R)}\,.$$

These in (3) give

$$n\cosh(a/R)\frac{\sinh(x/R)}{\sinh(p/R)} = n'\cosh(a'/R)\frac{\sinh(x'/R)}{\sinh(p/R)}\,. \tag{4}$$

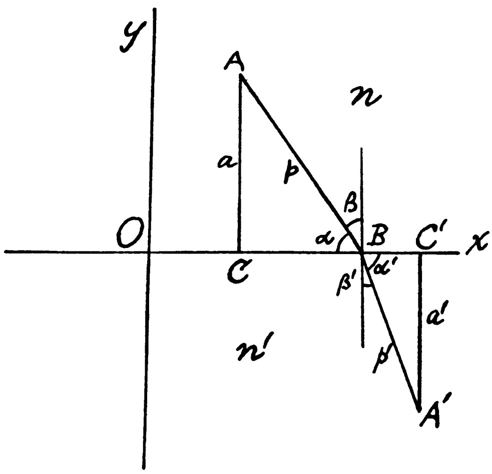

Fig. 2

But

$$\cos\alpha = \frac{\tanh(x/R)}{\sinh(p/R)}, \qquad \cos\alpha' = \frac{\tanh(x'/R)}{\sinh(p'/R)}\,.$$

These in (4) give

$$n\cosh(a/R)\frac{\cos\alpha}{\cosh(a/R)} = n'\cosh(a'/R)\frac{\cos\alpha'}{\cosh(a'/R)},$$

therefore

$$n\cos\alpha = n'\cos\alpha' \quad \text{or} \quad \frac{\sin\beta}{\sin\beta'} = \frac{n'}{n}, \tag{5}$$

which is the law of sines as in e-geometry.

In case of reflection, $n=n'$; we find in similar manner that the angle of incidence equals the angle of reflection.

We see at once the truth of the following

THEOREM. *The image of an object obtained by reflection on a plane mirror is the congruent figure back of the mirror and at the same distance as the object in front of it.*

Let us consider now refraction, on a plane surface, say $z_1=0$.

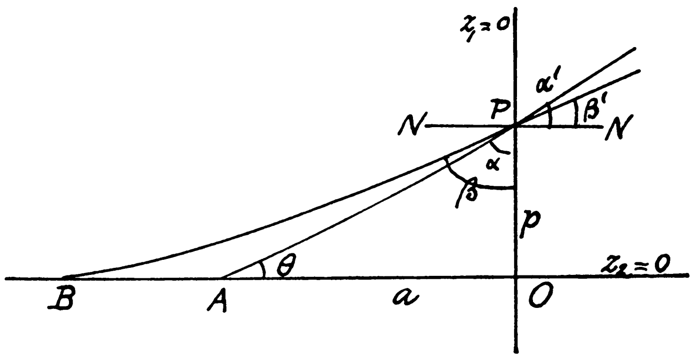

FIG. 3

In Fig. 3 a ray issues from A, and strikes the boundary surface at P, making the angle α' with the normal PN. The refracted ray PB' makes the angle β' with the normal, where

$$(6) \qquad \frac{\sin\,\beta'}{\sin\,\alpha'} = n < 1, \text{ say}.$$

Produced backwards, it cuts $z_2=0$ at B. We set

$$\alpha + \alpha' = 90^\circ, \quad \beta + \beta' = 90^\circ, \quad AO = a, \quad BO = b, \quad PO = p$$

in H-measure. Set also

$$C = \cosh\,(a/R), \quad S = \sinh\,(a/R), \quad T = \tanh\,(a/R).$$

Then

$$\tan\theta = \frac{\tanh\,(p/R)}{S}, \qquad \tan\alpha = \frac{T}{\sinh\,(p/R)}.$$

Thus

$$(7) \qquad \tan\alpha' = \frac{C\tan\theta}{(1 - S^2\tan^2\theta)^{1/2}},$$

while β' is given by (6).

Let us suppose θ is small; then (7) gives

$$\tan \alpha' = C\theta/(1 - S^2\theta^2)^{1/2} = C\theta, \text{ or } \alpha' = C\theta,$$

neglecting θ^3. Thus

$$\beta' = n\alpha' = nC\theta.$$

Next we have

$$\sinh (p/R) = \frac{S \tan \theta}{(1 - S^2 \tan^2 \theta)^{1/2}} = S\theta,$$

$$\tag{8} \tanh (b/R) = \frac{\sinh (p/R)}{\tan \beta'} = \frac{S \cdot \theta}{nC\theta} = \frac{1}{n} \tanh (a/R),$$

a constant, neglecting higher powers of θ.

Suppose now we revolve Fig. 3 about OA. The symmetry of the figure gives the following

THEOREM. *A nearly normal pencil of rays issuing from A forms a virtual image at B, at a distance b given by* (8).

Since $\tanh (b/R) \leqq 1$ we see that when a is such that

$$\tag{9} \tanh (a/R) \geqq n, \quad n < 1,$$

the ray PB does not meet the axis OA. Hence

THEOREM. *When the pencil enters a denser medium no image is formed if A is at a distance a satisfying* (9).

A small rotation θ about the vertex A_1 opposite the plane $z_1 = 0$ in Fig. 4 moves O to O' and LM to $L'M'$, such that $A'O' = AO$, $BO = B'O'$. As $A'O'$ is normal to the plane $z = 0$, we see the rays from A' behave in the same manner as those from A. Thus they meet in B'; hence

THEOREM. *The small figure AA' has BB' as its image.*

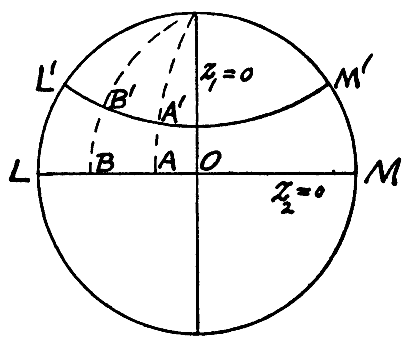

FIG. 4

Let δ be the distance of A' to LM and δ' the distance of B' to this line. Then

$$\sinh (\delta/R) = \cosh (\rho/R) \sin \theta, \quad \rho = OA'.$$

Similarly

$$\sinh(\delta'/R) = \cosh(\rho'/R)\sin\theta, \quad \rho' = OB'.$$

Hence

$$m = \frac{\sinh(\delta'/R)}{\sinh(\delta/R)} = \frac{\cosh(\rho'/R)}{\cosh(\rho/R)}.$$

This we may call the magnification of the image.

By (8)

$$\tanh(\rho'/R) = (1/n)\tanh(\rho/R).$$

Thus

$$m = \frac{n}{[n^2\cosh^2(\rho/R) - \sinh^2(\rho/R)]^{1/2}} = \frac{n}{[1-(1-n^2)\cosh^2(\rho/R)]^{1/2}}.$$

This becomes imaginary if

$$\frac{1}{1-n^2}\cosh^2(\rho/R) > 1, \text{ or } \tanh(\rho/R) > n.$$

It is not difficult to prove the

THEOREM. *If the pencil is not narrow, the rays issuing from A do not meet in a point.*

3. CENTRAL OPTICAL IMAGERY

As a first approximation to the path of light through a system of lenses the theory of collineation was shown by Maxwell and Abbe to be of extreme value. We shall now show that H-optics does not have this elegant tool to

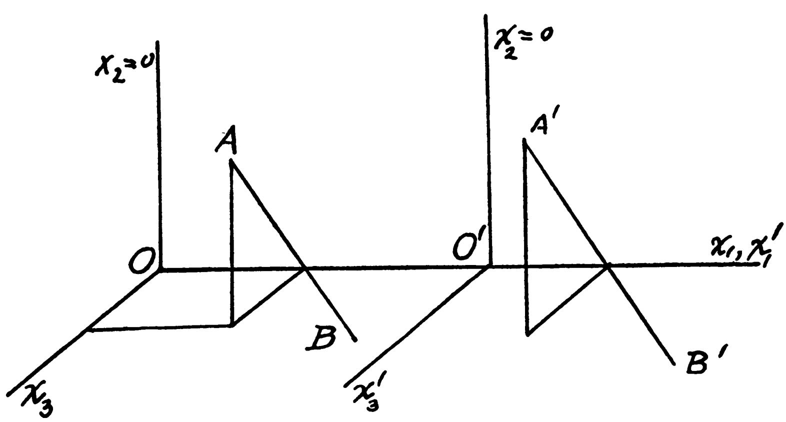

FIG. 5

work with. We suppose that we are dealing with a symmetrical optical system whose axis is the x_1-axis. The collineation defined by this system must have the form

$$
\begin{aligned}
z_1' &= a_{11}z_1 + a_{12}z_2 + a_{13}z_3 + a_{14}z_4, \\
&\cdots\cdots\cdots \\
z_4' &= a_{41}z_1 + a_{42}z_2 + a_{43}z_3 + a_{44}z_4.
\end{aligned} \tag{1}
$$

Let A, B be two symmetric points in the object space relative to the axis x_1, and A', B' their images. If the coördinates of A are (z_1, z_2, z_3, z_4), the coördinates of B are $(z_1, -z_2, -z_3, z_4)$ while the coördinates of B' are $(z_1', -z_2', -z_3', z_4')$. These in (1) give

$$
\begin{aligned}
z_1' &= a_{11}z_1 - a_{12}z_2 - a_{13}z_3 + a_{14}z_4, \\
-z_2' &= a_{21}z_1 - a_{22}z_2 - a_{23}z_3 + a_{24}z_4, \\
-z_3' &= a_{31}z_1 - a_{32}z_2 - a_{33}z_3 + a_{34}z_4, \\
z_4' &= a_{41}z_1 - a_{42}z_2 - a_{43}z_3 + a_{44}z_4.
\end{aligned} \tag{2}
$$

Comparing (1), (2) we get

$$a_{12} = a_{13} = 0, \quad a_{21} = a_{24} = 0, \quad a_{31} = a_{34} = 0, \quad a_{42} = a_{43} = 0.$$

Thus (1) reduces to

$$
z_1' = a_{11}z_1 + a_{14}z_4, \quad z_2' = a_{22}z_2 + a_{23}z_3, \quad z_3' = a_{32}z_2 + a_{33}z_3, \quad z_4' = a_{41}z_1 + a_{44}z_4. \tag{3}
$$

Since the collineation is central we may restrict ourselves to a plane, say the x_1x_2 plane. We may thus write (3)

$$
z_1' = az_1 + bz_3, \quad z_2' = cz_2, \quad z_3' = \alpha z_1 + \beta z_3. \tag{4}
$$

Since the coördinates z' must satisfy the relation §1, (5), we find that

$$
a^2 - \alpha^2 = 1, \quad b^2 - \beta^2 = -1, \quad c^2 = 1, \quad ab = \alpha\beta. \tag{5}
$$

Hence

$$
z_1' = az_1 + bz_3, \quad z_2' = \delta z_2, \quad z_3' = \epsilon bz_1 + \epsilon az_3, \tag{6}
$$

where $\delta^2 = \epsilon^2 = 1$.

Now the equations (6) define a displacement , i.e. the image is merely a congruent figure of the object. Hence

THEOREM. *If the image afforded by a central optical system is the result of a collineation, the image is an exact replica of the object without magnification.*

If the coördinates of an ideal point are set in (6) we find $\{z'^2\}=0$; hence

THEOREM. *As a point A recedes to infinity, its image A' does the same.*

4. REFLECTION ON A SPHERE

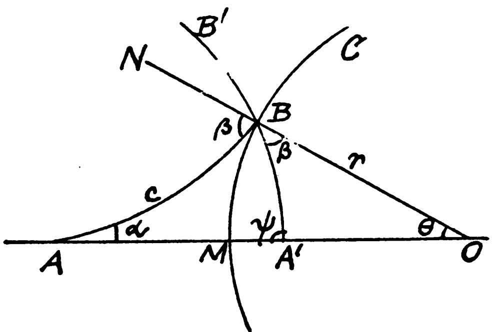

FIG. 6

If a ray issuing from A meets the spherical mirror C at a point B, it is reflected along BB':

$$OA = a, \quad OB = r, \quad AB = c, \quad OA' = a',$$

in H-measure. In the triangle OAB,

$$\cosh (c/R) = \cosh (a/R) \cosh (r/R) - \sinh (a/R) \sinh (r/R) \cos\theta, \tag{1}$$

$$\sin\beta = \frac{\sinh (a/R)}{\sinh (c/R)} \sin\theta. \tag{2}$$

In the triangle OBA',

$$\cos\psi = \cos\beta\cos\theta - \sin\beta\sin\theta\cosh (r/R), \tag{3}$$

$$\sinh (a'/R) = \frac{\sin\beta}{\sin\psi} \sinh (r/R). \tag{4}$$

These equations give the path of the reflected ray.

The H-length of the arc $MB=l=R\theta \sinh (r/R)$. We set $l\leqq l_0$, $\theta\leqq\theta_0$, and suppose θ_0 and l_0/R are small. We will assume that r/R is small while a/R is large. Then approximately

$$\cosh (a/R) = \sinh (a/R) = \tfrac{1}{2}e^{a/R}.$$

Then (1) gives

$$\cosh (c/R) = \tfrac{1}{2}e^{a/R} = \cosh (a/R) \text{ ; therefore } a = c.$$

This in (2) gives

$$\beta = \theta.$$

Thus (3) gives

$$\cos\psi = \cos^2\theta - \sin^2\theta\cosh (r/R)$$
$$= 1 - \theta^2 - \theta^2\left(1+\frac{r^2}{2R^2}\right) = 1 - 2\theta^2 = 1 - \frac{(2\theta)^2}{2}.$$

Hence

$$\sin\psi = 2\theta, \quad \text{or} \quad \psi = 2\theta.$$

This in (4) gives

$$\sinh(a'/R) = \frac{\theta}{2\theta}\sinh(r/R) = \tfrac{1}{2}(r/R),$$

therefore

$$a'/R = \tfrac{1}{2}(r/R) \quad \text{or} \quad a = \tfrac{1}{2}r.$$

Hence rays issuing from A meet at a fixed point A'.

Let us now rotate Fig. 6 through a small angle about O. The point A describes the arc AP while its virtual image describes the arc $A'P'$. Hence

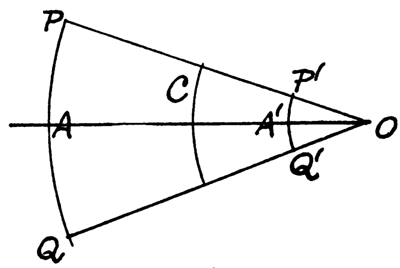

FIG. 7

THEOREM. *The image $P'Q'$ of a small distant object PQ in a small mirror of radius r, small compared with the space constant R, is virtual and lies at the distance $\frac{1}{2}r$ from the center of the mirror.*

5. REFRACTION ON SPHERICAL SURFACES

In Fig. 8, a ray issues from O meeting the spherical surface at B and makes the angle α with the normal BN. It is bent into BH which produced

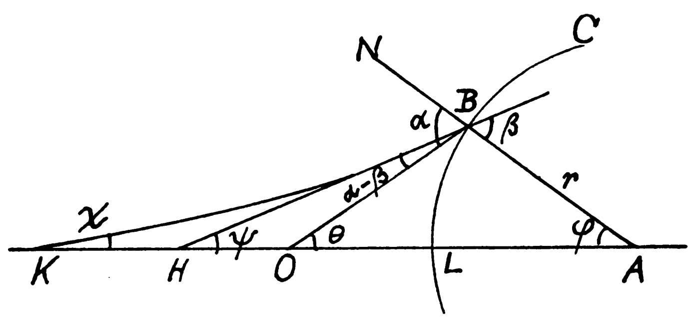

FIG. 8

backwards meets OA at K. Its tangent at B is BH:

$$OA = a, \quad OB = s, \quad OH = h, \quad \text{rad}\,AB = r, \quad \text{in } e\text{-measure};$$

$$OK = \eta, \quad OB = \sigma, \quad OA = \omega, \text{ in } H\text{-measure}.$$

Then

$$\tan\theta = \frac{r\sin\phi}{a - r\cos\phi}, \quad \theta = \beta + \psi = \alpha - \beta + \psi, \tag{1}$$

$$\sin\beta = n\sin\alpha,\ n < 1,\ \alpha = \theta + \phi, \tag{2}$$

$$s = r\frac{\sin\phi}{\sin\theta}, \qquad h = \frac{s\sin(\alpha - \beta)}{\sin\psi}. \tag{3}$$

In the H-triangle OBK,

$$\cos\chi = \cos(\alpha - \beta)\cos\theta + C\sin(\alpha - \beta)\cos\theta, \tag{4}$$

$$\sinh(\eta/R) = \frac{\sin(\alpha - \beta)}{\sin\chi}\cdot S, \tag{5}$$

where

$$C = \cosh(\sigma/R), \qquad S = \sinh(\sigma/R). \tag{6}$$

These equations give the refracted ray.

We suppose now that ϕ is small. From (1) we have, neglecting small quantities of higher order,

$$\theta = \frac{r\phi}{a - r}. \tag{7}$$

From (2),

$$\alpha = \theta + \phi = \frac{a\phi}{a - r}, \tag{8}$$

$$\alpha/\theta = a/r, \quad \text{a constant}. \tag{9}$$

From (2),

$$\beta = n\alpha = \frac{na\phi}{a - r}, \quad \alpha - \beta = (1 - n)\alpha = m\alpha, \quad m = 1 - n. \tag{10}$$

From (3),

$$s = r\phi/\theta = a - r. \tag{11}$$

Thus s and hence σ, its H-measure, is constant. From (4),

$$\cos\chi = 1 - \tfrac{1}{2}(m^2\alpha^2 + \theta^2 - 2m\alpha\theta C) = 1 - \tfrac{1}{2}\theta^2X^2,$$

where

$$X^2 = m^2(\alpha^2/\theta^2) + 1 - 2mC(\alpha/\theta);$$

therefore, using (9),

$$X^2 = m^2(a^2/r^2) + 1 - 2mC(a/r), \quad \text{a constant.} \tag{12}$$

Thus

$$\sin \chi = \theta \cdot X.$$

From (9), (10) and (15)

$$\sinh (\eta/R) = \frac{(\alpha - \beta)S}{\theta X} = \frac{ma}{r}\,\frac{S}{X}, \quad \text{a constant.} \tag{13}$$

Hence

THEOREM. *Neglecting small quantities of order* >1, *rays issuing from* O *and meeting a convex spherical surface nearly normally, meet at the virtual image* K *at a distance* η *given by* (13)

For χ to be real, X must $\geqq 0$, or

$$C = \cosh (\sigma/R) \leqq \frac{r^2 + m^2a^2}{2amr}. \tag{14}$$

Now as $a \to 2R$, $\sigma \to \infty$ while the right side is finite, we have the

THEOREM. *When the source moves away beyond a certain distance given by* (14) *there is no image.*

We note that when the = sign holds in (14) the point K is at ∞.

Let us displace Fig. 8 by a rotation about the pole of OA, so that A coincides with the origin O. Let the source be now at A, Fig. 9, and its image at C.

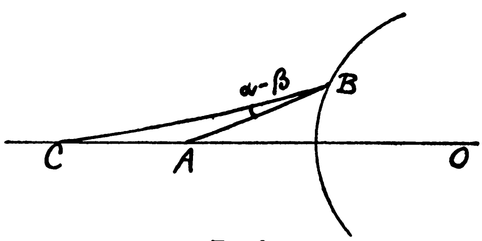

FIG. 9

Since distances and angles have remained unaltered we have as before $AB = \sigma$, $OK = \eta$, and relation (5) still holds.

Let us now revolve Fig. 9 about O through a small angle τ. Then A and C describe small arcs AA', CC' of lengths

$$\delta = R\tau \sinh (\omega/R), \quad \delta' = R\tau \sinh [(\omega + \eta)/R].$$

We may regard as the magnification of the object the quotient

$$\frac{\delta'}{\delta} = \frac{\sinh [(\omega + \eta)/R]}{\sinh (\omega/R)}. \tag{15}$$

Finally let us note that the point H is fixed. For from (3)

$$h = \frac{(\alpha - \beta)s}{\psi} = \frac{ms\alpha}{\psi} = \frac{ms\alpha}{\theta - m\alpha},$$

which is constant since α/θ is, by (9). We note that when the source of light recedes to infinity the point H is real.

6. Lenses

In practical optics when one wishes to calculate the best shapes of a new system of lenses designed to achieve a certain object, it is necessary in the end to make laborious trigonometric calculations. In H-space it seems necessary to do this even when one wishes only approximate results since the theory of collineation does not apply. As in the preceding sections, so here, we make a favorable choice of the origin, sometimes using e-measure and sometimes H-measure, angles having the same measure in both geometries.

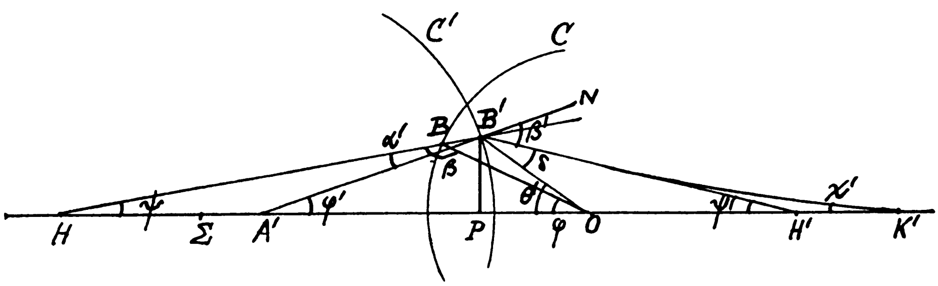

Fig. 10

In Fig. 10 we suppose the center of the lense C is at the origin O while the center of C' is at A'. The source of light is at Σ. A ray meets C at B and is bent into BB', which as before we may suppose to be an e-straight. At B it is bent into $B'K'$ whose e-tangent at B' is $B'H'$. The incident ray at B' makes the angle α', the emergent ray makes the angle β' with the normal $B'N'$ at B'. Then

$$\sin\beta' = \frac{1}{n}\sin\alpha'. \tag{1}$$

We set

$$\beta = OBH,\ \psi = B'HA',\ \phi' = B'A'O,\ \phi = BOA',\ \theta' = B'OA',$$
$$\psi' = B'H'O,\ \chi' = B'K'O;$$

$$OH = h, \ OH' = h', \ OA' = a', \ OB' = s', \ l = A'H = h - a',$$
$$OB = r, \ A'B' = r', \text{ in } e\text{–measure};$$
$$\sigma' = OB', \quad \eta' = OK' \text{ in } H\text{–measure};$$
$$C' = \cosh(\sigma'/R), \qquad S' = \sinh(\sigma'/R).$$

We saw in §5 that H is fixed when ϕ is small. We have now

$$\sin\alpha' = (l/r')\sin\psi, \tag{2}$$

$$\tan\theta' = \frac{r'\sin\phi'}{a' - r'\cos\phi'}, \tag{3}$$

$$s' = \frac{r'\sin\phi'}{\sin\theta'}. \tag{4}$$

In the H-triangle $OB'K'$, $\delta = OB'K' = \theta' + \phi' - \beta'$, and

$$\cos\chi' = \cos\delta\cos\theta' + C'\sin\delta\sin\theta', \tag{5}$$

$$\sinh(\eta'/R) = S'\frac{\sin\delta}{\sin\chi'}. \tag{6}$$

Suppose again that ϕ is small. Then

$$\psi = \beta - \phi = n\alpha - \phi, \quad \alpha' = l\psi/r', \tag{7}$$

$$\phi' = \alpha' + \psi = (l + r')\psi/r', \tag{8}$$

$$\theta = \frac{r'\phi'}{a' - r'} = \frac{l + r'}{a' - r'}\psi = g\psi, \quad g \text{ constant}, \tag{9}$$

$$s' = r'\phi'/\theta' = (a' - r')/r', \quad \text{a constant}. \tag{10}$$

Thus neglecting small quantities of order >1, s' and hence σ', also C', S' are constants. From (5),

$$\cos\chi' = (1 - \delta^2/2)(1 - \theta'^2/2) + 2C'\delta\theta'$$
$$= 1 - \tfrac{1}{2}\theta'^2(1 + \delta^2/\theta'^2 - 2C'\delta/\theta') = 1 - \tfrac{1}{2}\theta'^2X^2,$$

therefore

$$\sin\chi' = \theta'X, \quad \chi' = \theta'X.$$

From (6)

$$\sinh(\eta'/R) = \frac{\delta}{\theta'}\cdot\frac{S'}{X}. \tag{11}$$

Now

$$\delta/\theta' = 1 + \phi'/\theta' - \beta'/\theta', \quad \beta' = \frac{\alpha'}{n} = \frac{l\psi}{nr'};$$

therefore

$$\beta'/\theta' = \frac{l}{nr'} \cdot \frac{\psi}{\theta'} = \frac{l}{gnr'}, \quad \text{a constant},$$

whence δ/θ' is constant, and therefore by (11), η' is constant.

In order that χ' be real, we must have $X^2 \geqq 0$ or

$$(\delta/\theta' - C')^2 \geqq C'^2 - 1 = S'^2 \text{ or } \delta/\theta' \geqq C' \pm S' = \Delta = e^{\pm\sigma'/R},$$

or

$$a' - \frac{l}{gn} \geqq \Delta \text{ or } na' - \frac{a' - r'}{1 - r'/l} \geqq n\Delta. \tag{12}$$

When r'/l can be neglected, this gives

$$r' \geqq (1 - n)a' + n\Delta. \tag{13}$$

Hence the

THEOREM. *If the conditions* (12) *or* (13) *are satisfied, rays issuing from a point and meeting a convex lens nearly normally will unite in a conjugate point K' determined by* (11).

7. Bouguer's theorem

We suppose that the index of refraction n at a point $x_1x_2x_3$ is a function of the distance of the point from the origin O. The path of a ray in this medium is determined by

$$\delta \int n\, ds = 0.$$

We have

$$\delta(n\, ds) = n \cdot \delta\, ds + ds \cdot \delta n, \qquad \delta n = \sum_\alpha \frac{\partial n}{\partial x_\alpha} \delta x_\alpha,$$

$$\delta \cdot ds = \frac{16R^2}{\lambda^2} \sum_\alpha \frac{dx_\alpha}{ds} \delta \cdot dx_\alpha + \frac{2}{\lambda^2} \sum_\alpha x_\alpha \delta x_\alpha,$$

$$\int_a^b \frac{n}{\lambda^2} \frac{dx_\alpha}{ds} \delta \cdot dx_\alpha = -\int_a^b \delta x_\alpha \frac{d}{ds}\left(\frac{n}{\lambda^2} \frac{dx_\alpha}{ds}\right) ds,$$

therefore

$$\delta \int_a^b n\, ds = \int_a^b ds \sum_\alpha \left\{ \frac{\partial n}{\partial x_\alpha} + \frac{2nx_\alpha}{\lambda^2} - 16R^2 \frac{d}{ds}\left(\frac{n}{\lambda^2} \frac{dx_\alpha}{ds}\right) \right\} \delta x_\alpha.$$

Thus the equations of the path are

$$\frac{\partial n}{\partial x_\alpha}+\frac{2nx_\alpha}{\lambda^2}-16R^2\frac{d}{ds}\left(\frac{n}{\lambda^2}\frac{dx_\alpha}{ds}\right)=0 \qquad (\alpha=1,2,3). \tag{1}$$

The matrix

$$\begin{pmatrix} x_1, & x_2, & x_3 \\ \dfrac{dx_1}{ds}, & \dfrac{dx_2}{ds}, & \dfrac{dx_3}{ds} \end{pmatrix} \tag{2}$$

has three determinants D_α. From (1) we find at once that

$$\frac{d}{ds}\left(\frac{n}{\lambda^2}\cdot D_\alpha\right)=0.$$

Hence

$$\frac{n}{\lambda^2}\cdot D_\alpha=a_\alpha,\quad \text{a constant}.$$

If we multiply these three equations by x_1, x_2, x_3 we get $a_1x_1+a_2x_2+a_3x_3=0$. Hence

THEOREM. *The path of a ray in this medium lies in an H-plane through O.*

In our model, the radius vector r from O to the point $P(x_1, x_2, x_3)$ has as direction cosines $l_\alpha=x_\alpha/r$, while those of the ray are $m_\alpha=dx_\alpha/d\sigma$. If θ is the angle between r and the ray,

$$\cos\theta=\sum l_\alpha m_\alpha.$$

Now

$$\frac{dx_\alpha}{ds}=\frac{\lambda}{4R}\frac{dx_\alpha}{d\sigma}=\frac{\lambda m_\alpha}{4R},$$

therefore

$$\cos\theta=\frac{4R}{r\lambda}\sum x_\alpha\frac{dx_\alpha}{ds}.$$

Here the sum on the right is the scalar product of the matrix (2). By Lagrange's theorem,

$$\sum D_\alpha^2=\sum x_\alpha^2\cdot\sum\frac{dx_\alpha^2}{ds^2}-\left(\sum x_\alpha\frac{dx_\alpha}{ds}\right)^2$$

or

$$\frac{\lambda^4}{n^2}\sum a_\alpha^2=r^2\frac{\lambda^2}{16R^2}(1-\cos^2\theta)=r^2\frac{\lambda^2}{16R^2}\sin^2\theta.$$

Hence

$$\frac{nr}{\lambda}\sin\theta = \text{constant along the ray}. \tag{3}$$

This gives Bouguer's theorem in H_3-space.

THEOREM. *The path of a ray of light in a medium whose index is a function only of the distance from O satisfies* (3).

If ρ is the length of the vector OP in H-measure we have

$$r = 2R\tanh(\rho/2R), \quad \lambda = 4R^2\operatorname{sech}^2(\rho/2R), \quad \frac{r}{\lambda} = \frac{\sinh(\rho/R)}{4R}.$$

Thus (3) becomes

$$n\sinh(\rho/R)\sin\theta = c, \quad \text{a constant}, \tag{4}$$

where the quantities ρ, θ, η are now expressed in H-measure. From this relation we get the equation of the path of the ray. For

$$d\rho = \cos\theta\, ds, \qquad d\phi = \frac{(\sin\theta)\, ds}{R\sinh(\rho/R)},$$

therefore

$$\frac{d\rho}{d\phi} = \frac{\cos\theta}{\sin\theta}\cdot R\sinh(\rho/R),$$

or using (4)

$$\frac{d\phi}{d\rho} = \frac{1}{(n^2\sinh^2(\rho/R) - c^2)^{1/2}}. \tag{5}$$

Here n is a function of ρ only.

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